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Approximate solution of 2D Neumann problem in arbitrary simply connected domains with smooth boundaries

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Аспирант

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We present a method for construction of continuous approximate 2d Neumann problem solution in an arbitrary simply connected domain with a smooth boundary. The method reduces the problem to the corresponding Dirichlet problem, then the Cauchy integral Method is applied.

Let Ω be a simply connected domain, $\partial\Omega = \{(x(t), y(t)), t \in [0, 2\pi]\}$ be the smooth boundary of Ω. Then the corresponding Neumann problem for the Laplace equation is to find the doubly differentiable in Ω function $u(x, y)$, which is continuous in $\Omega \cup \partial\Omega$ and satisfies the Laplace equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0, \qquad (x,y) \in \Omega, \tag{1}$$

and the Neumann boundary conditions defined in the parametric form:

$$\frac{\partial u}{\partial n} = f(t), \qquad \text{on } \partial\Omega, \qquad t \in [0, 2\pi], \tag{2}$$

where n is a derivative in the outward direction normal to the boundary of Ω. The algorithm of solution as follows:
let $v(x, y)$ be the conjugate harmonic to $u(x, y)$ in Ω. By using the Cauchy-Riemann conditions we can deduce that $\frac{\partial u}{\partial n} = \frac{\partial v}{\partial s}$, where s is the direction tangent to the boundary $\partial\Omega$. Now the Dirichlet problem is to be solved: find the harmonic in Ω function $v(x, y)$ with the corresponding boundary condition:

$$v(x(t), y(t)) = \int_0^t f(t)|z'(t)|dt, \qquad \text{on } \partial\Omega, \qquad t \in [0, 2\pi], \tag{3}$$

where $z(t) = x(t) + iy(t)$. Evidently $v\left(x(0), y(0)\right) = v\left(x(2\pi), y(2\pi)\right)$, so we can write the necessary condition of solvability of this problem as follows:

$$\int_0^{2\pi} f(t)|z'(t)|dt = 0,$$

and this agrees with the necessary condition of solvability of Neumann problem.

The obtained Dirichlet problem is solved using the Cauchy integral method which discussed in details in our previous work [1]. We restore the boundary values of the imaginary part of an analytic in Ω function $B(z)$ via the boundary values of the real part of this function. The method constructs the analytic function $B(z)$ in Ω by solving a Fredholm integral equation of second type, then the harmonic solution is written as $v(x, y) = \mathrm{Re}(B(z))$, $z = (x+iy) \in \Omega$. The final solution of Neumann problem is obtained with the help of Cauchy-Riemann conditions of an analytic function. We obtain the solution of Neumann problem within arbitrary summand.

Due to the singularity of the Cauchy integral at the points near the boundary of Ω, we apply the Taylor series analytic continuation for these points [2].

The proposed method was applied to several examples and it gave accurate results. Fig(1) shows the comparison between the numerical and exact solution of the Neumann problem for Laplace equation in the simply connected domain with boundary curve:

$z(t) = r(t)\,[\cos(t) + i\sin(t)], r(t) = \sqrt{(a+b)^2 + 1 - 2(a+b)\cos\frac{at}{b}},\ t \in [0, 2\pi]\,.$

The Cauchy integral method gives accurate results for the solution of 2D Neumann problem for irregular simply connected domains. The comparison between exact and approximate solution gave maximum absolute error less than 10E-4. The method is applicable for domains bounded by any smooth curve approximated by Fourier polynomial.

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Источники и литература

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Иллюстрации

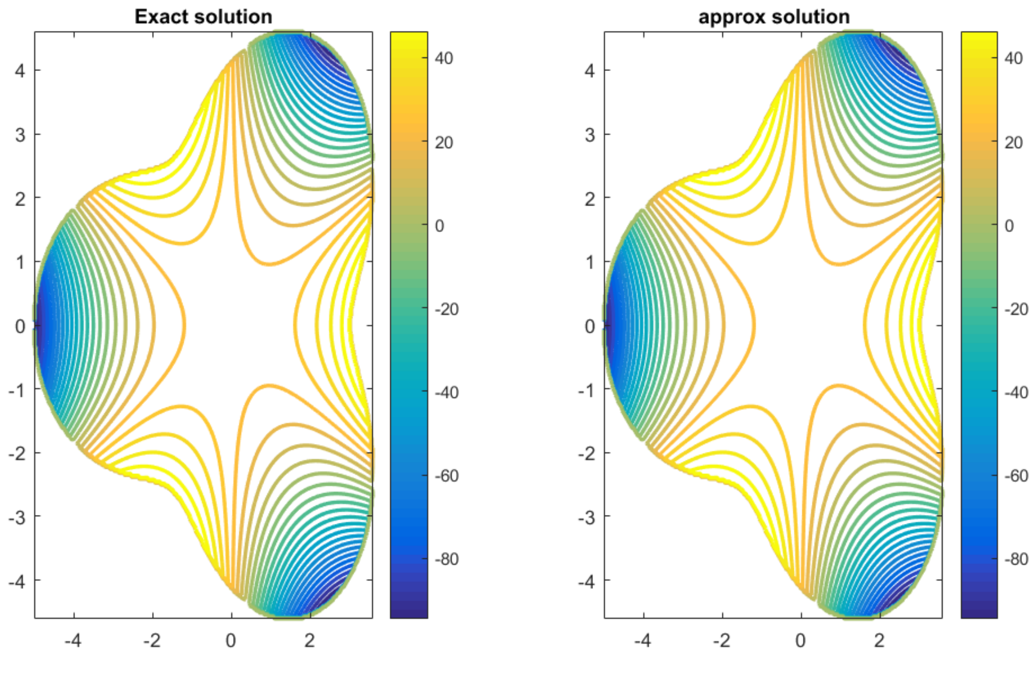

Рис. 1. Comparison between the contour plots of the numerical and exact solutions with $a = 4, b = 1$ by using the exact solution $u(x, y) = x^3 - 3xy^2$.